$$M = 25, \qquad n_2^\times = 16, \qquad n_3^\times = 1$$

and according to relation (2.2)

$$n_p^\times = 1 \cdot n_2^\times + 2n_3^\times = 1 \cdot 16 + 2 \cdot 1 = 18$$

$$\gamma = 1,\ \gamma' = 2,\ n_p^\circ = 24,\ S_0 = 18,\ n_0 = 12,\ \min \rho_0 = 25,\ \min d = 19$$

The reduced number of gaps M_p given by relation (2.8) is

$$M_p = 25 - 18 - 1 = 18 - 12 = 24 - 18 = {}^1/_2\,(24 - 12) = 25 - 19 = 6$$

According to relation (2.13) the number of faces is

$$\Gamma = M_p + 2\gamma' = 6 + 4 = 10$$

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THE INSTANT OF FORMATION OF A SHOCK WAVE IN A TWO-WAY TRAFFIC FLOW

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G. P. SOLDATOV
(Saratov)

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Two continuity equations describing a symmetric two-way traffic flow are considered. These equations are then used to find the instant of formation of a shock wave by the Riemann method.

The theoretical analysis of traffic flows has lately received much attention; theories of traffic flows have been constructed on the basis of analysis of motion of descrete objects (often called "motorcars"), mathematical statistics [1, 2], classical mechanics [3–5], and statistical mechanics [6]. Survey [5] contains a discussion of studies applying the hydrodynamic analogy to traffic flows. The first of the two current trends of research is based on the kinematic wave theory; the second is based on the Greenberg relation for continuous traffic flows. The kinematic waves traveling opposite to the direction of traffic and the distinctions between them and dynamic waves are considered in [3].

We make use of the hydrodynamic model of a two-way traffic flow. The traffic flow in this model is described by two continuity equations and by two empirical relationships between the velocities and densities of the flows of cars moving in opposite directions. The hydrodynamic model of a traffic flow enables us to predict the formation of shock waves and to analyze many cases of shock wave propagation in the flow. The basic properties of traffic flow are established in [4].

The propagation of a wave over the homogeneous flow range is considered under the assumption that the initial flow density distribution has a discontinuity in its derivative with respect to the direction, so that the wave front intersects like characteristics. The instant t_* at which an indeterminacy domain arises is computed by mapping the flow onto itself [7], whereupon the points of intersection of the characteristics of the same family turn out to be critical points of the mapping.

1. The two-way flow equations. Let p and q be the densities of the car flow and u and v the average velocities of the cars moving to the right and to the left, respectively, so that $\varphi = pu \geqslant 0, \quad u \geqslant 0; \qquad \Psi = qv \leqslant 0, \quad v \leqslant 0$

A two-way traffic flow is described by the continuity equations

$$p_t + \varphi_x = 0, \qquad q_t + \Psi_x = 0 \tag{1.1}$$

and by the functions $u = f(p, q)$, $v = g(p, q)$ which are assumed to be known from experiment [4, 5]. The functions f and g have the following properties:

f is a monotonically decreasing function, g is a monotonically increasing function;
for every q (P) there is value of P (q) for which $u = 0$ $(v = 0)$;
$f(p, q) = -g(p, q)$ is the condition of symmetry of the flows.

In the hodograph plane Eqs. (1.1) become

$$x_q - (pu)_p t_q + pu_q t_p = 0, \quad x_p - (qv)_q t_p + qv_p t_q = 0$$

Let us rewrite Eqs. (1.1) in matrix form,

$$U_t + AU_x + B = 0$$

$$U = \begin{Bmatrix} p \\ q \end{Bmatrix}, \qquad A = \begin{pmatrix} (pu)_p & pu_q \\ qV_p & (qV)_q \end{pmatrix}, \qquad B \equiv 0$$

Let $\lambda^{1,2}$ be an eigenvalue and $L^{1,2}$ an eigenvector of the matrix A. Then

$$\lambda^{1,2} = {}^1/_2\,[(pu)_p + (qv)_q] \pm {}^1/_2\, R$$
$$R^2 = [(pu)_p - (qv)_q]^2 + 4_{pq}\, u_q v_p$$
$$L^{1,2} = ([(pu)_p - (qv)_q] \pm R \quad 2pu_q)$$

Moreover, in the hodograph plane

$$\mu^{1,2} = \frac{dq}{dp} = \frac{-[(pu)_p - (qv)_q] \pm R}{2_p u_q}$$

2. Expressions for the critical instant t_* [7]. Let $\Phi(t, x)$ be the wave front. We introduce the coordinates t' and Φ, setting

$$t' = t, \quad \Phi_t + \lambda^\Phi \Phi_x = 0$$

where λ^Φ is the eigenvalue corresponding to the wave front. Then, multiplying the matrix equation by the vector L^i, we obtain

$$L^i \left(x_\Phi \frac{\partial}{\partial t'} + (\lambda^i - \lambda^\Phi) \frac{\partial}{\partial \Phi}\right) U = 0$$
$$L^\Phi U_{t'} = 0, \qquad \text{if } \lambda^i = \lambda^\Phi$$

Let

$$\Pi = [U_\Phi]_+^- = U_\Phi \Big|_{\Phi=0_-} - U_\Phi \Big|_{\Phi=0_+} \qquad X = [x_\Phi]_+^-$$

and let the subscript 0 refer to the constant conditions ahead of the wave front. The foregoing conditions then imply that

$$L_0^i\Pi = 0, \quad \lambda^i \neq \lambda^\Phi; \quad L_0^\Phi \Pi_{t'} = 0, \quad \lambda^i = \lambda^\Phi, \quad i = \Phi = 1 \tag{2.1}$$

The solution of Eqs. (2.1) can be written as

$$\Pi = \begin{Bmatrix} \Pi_1 \\ \Pi_2 \end{Bmatrix} = \Pi_1^* \begin{Bmatrix} 1 \\ \mu_1 \end{Bmatrix} \qquad (\Pi_1^* = \lim_{t\to 0} \Pi_1)$$

Let us consider the expression

$$\frac{\partial}{\partial t'} x_\Phi = (\nabla_u \lambda^\Phi)\, U_\Phi$$

where ∇_u is a gradient operator in U-space. Along $\Phi = 0$ we have

$$X_{t'} = (\nabla_u \; \lambda^\Phi)_0 \, \Pi$$

or

$$X = X^* + \int_0^t (\nabla_u \; \lambda^\Phi)_0 \Pi \, dt'$$

The instant t_* is therefore determined by the relation

$$0 = x_\Phi^* + \int_0^{t_*} (\nabla_u \lambda^\Phi)_0 \, \Pi \, dt' \tag{2.2}$$

Substituting

$$(\nabla_u \lambda^\Phi)_0 = \Pi_1^* \left(\frac{\partial \lambda^1}{\partial p} + \mu^1 \frac{\partial \lambda^1}{\partial q} \right)$$

into expression (2.2), we obtain

$$\frac{1}{t_*} = -p_x^* \left(\frac{\partial \lambda^1}{\partial p} + \mu^1 \frac{\partial \lambda^1}{\partial q} \right)$$

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